4.6 ทฤษฎีบทไดเวอร์เจนซ์ (The Divergence Theorem)

ในส่วนแรกนี้เราจะพิจารณาตัวดำเนินการ (operator) ที่มีความสำคัญใน 3D คือ ไดเวอร์เจนซ์ (divergence) และเคิร์ล (curl)

ตัวดำเนินการไดเวอร์เจนซ์นี้เกี่ยวข้องกับการไหลของของไหลซึ่งไหลออกไปจากจุดหนึ่งๆในปริภูมิ ส่วนเคิร์ลเกี่ยวข้องกับการหมุนของของไหลรอบจุดในปริภูมิ

<u>นิยาม</u>: ให้ $\mathbb{F}(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ เราจะนิยามไดเวอร์เจนซ์ของ $\mathbb{F}$ ซึ่งเขียนแทนด้วย div $\mathbb{F}$ เป็น

$$\operatorname{div} \mathbb{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

เราจะนิยามเคิร์ลของ $\mathbb{F}$ ซึ่งเขียนแทนด้วย curl $\mathbb{F}$ เป็น

$$\operatorname{curl} \mathbb{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

ตัวอย่าง : จงหาไดเวอร์เจนซ์และเคิร์ลของ

$$\mathbb{F}(x,y,z) = x^2 y i + 2y^3 z j + 3z k$$

วิธีทำ จก $\mathbb{F}(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$

$$\Rightarrow \operatorname{div}\mathbb{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2xy + 6y^2 z + 3$$

และ

$$\operatorname{curl}\mathbb{F} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 y & 2y^2 z & 3z & x^2 y & 2y^2 z \end{array}\right.$$

$$= \frac{\partial 3z}{\partial y} i + \frac{\partial x^2 y}{\partial z} j + \frac{\partial 2y^2 z}{\partial x} k$$

$$- \frac{\partial x^2 y}{\partial y} k - \frac{\partial 2y^2 z}{\partial z} i - \frac{\partial 3z}{\partial x} j$$

$$= 0 \cdot i + 0 \cdot j + 0 \cdot k - x^2 k - 2y^2 i + 0 \cdot j$$

$$= -2y^2 i - x^2 k$$

$\square$

ตัวอย่าง: จงแสดงว่าไดเวอร์เจนซ์ของ

$$\mathbb{F}(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{3/2}}(xi+yj+zk)$$

มีค่าเป็นศูนย์

วิธีทำ

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{cx}{(x^2+y^2+z^2)^{3/2}}\right] =$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left[\frac{cy}{(x^2+y^2+z^2)^{3/2}}\right] =$$

$$\frac{\partial h}{\partial z} = \frac{\partial}{\partial z}\left[\frac{cz}{(x^2+y^2+z^2)^{3/2}}\right] =$$

$$\Rightarrow \operatorname{div} \mathbb{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

ผลลัพธ์ข้างล่างนี้เป็นทฤษฎีบทไดเวอร์เจนซ์ หรือทฤษฎีบทของเกาส์ ซึ่งเป็นการเขียนปริพันธ์สามชั้นบนรูปทรงต่างๆ ในเทอมของปริพันธ์ตามผิวบนผิวที่เป็นขอบของรูปทรงนั้น

ทฤษฎีบท 4.6.1 ให้ G เป็นรูปทรงที่หุ้มด้วยผิว σ ที่กำหนดทิศทางด้วยเวกเตอร์แนวฉากหนึ่งหน่วยด้านนอก ถ้า

$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$$

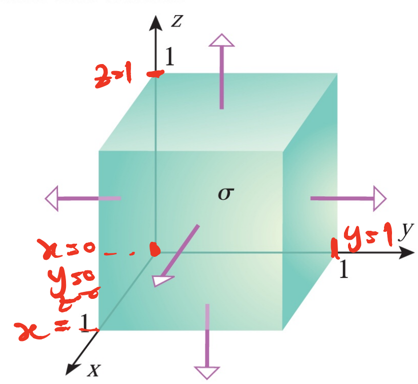
โดยที่ f, g และ h มีอนุพันธ์ย่อยอันดับหนึ่งที่ต่อเนื่องบนบริเวณเปิดที่บรรจุ G แล้ว

$$\iint_{\sigma} \vec{F}\cdot\vec{n}\,dS = \iiint_{G} div\,\vec{F}\,dV$$

ตัวอย่าง 4.6.4 กำหนดให้ G เป็นลูกบาศก์ในอัฐภาคที่หนึ่ง ดังแสดงในรูปที่ 18.6.2 และให้ σ เป็นผิวของลูกบาศก์ จงใช้ทฤษฎีบทไดเวอร์เจนซ์หาค่า

$$\iint_{\sigma} \vec{F}\cdot\vec{n}\,dS$$

เมื่อ $\vec{F}(x,y,z) = 2x\vec{i} + 3y\vec{j} + z^2\vec{k}$ และ $\vec{n}$ เป็นเวกเตอร์แนวฉากหนึ่งหน่วยด้านนอก



วิธีทำ $$\iint_{\sigma} \mathbb{F}\cdot n\,dS = \iiint_{G} div\,\mathbb{F}\,dV$$

$$div\,\mathbb{F} = div\left[2xi + 3yj + z^2k\right]$$

$$= \frac{\partial[2x]}{\partial x} + \frac{\partial[3y]}{\partial y} + \frac{\partial[z^2]}{\partial z}$$

$$= 2 + 3 + 2z \quad = 5 + 2z$$

$$\iint_{\sigma} \mathbb{F}\cdot n\,dS = \iiint_{G} div\,\mathbb{F}\,dv$$

$$= \int_{z=0}^{z=1}\int_{y=0}^{y=1}\int_{x=0}^{x=1} (5+2z)\,dx\,dy\,dz$$

□

Carl Friedrich Gauss (1777–1855) German mathematician and scientist. Sometimes called the "prince of mathematicians," Gauss ranks with Newton and Archimedes as one of the three greatest mathematicians who ever lived. His father, a laborer, was an uncouth but honest man who would have liked Gauss to take up a trade such as gardening or bricklaying; but the boy's genius for mathematics was not to be denied. In the entire history of mathematics there may never have been a child so precocious as Gauss—by his own account he worked out the rudiments of arithmetic before he could talk. One day, before he was even three years old, his genius became apparent to his parents in a very dramatic way. His father was preparing the weekly payroll for the laborers under his charge while the boy watched quietly from a corner. At the end of the long and tedious calculation, Gauss informed his father that there was an error in the result and stated the answer, which he had worked out in his head. To the astonishment of his parents, a check of the computations showed Gauss to be correct!

For his elementary education Gauss was enrolled in a squalid school run by a man named Büttner whose main teaching technique was thrashing. Büttner was in the habit of assigning long addition problems which, unknown to his students, were arithmetic progressions that he could sum up using formulas. On the first day that Gauss entered the arithmetic class, the students were asked to sum the numbers from 1 to 100. But no sooner had Büttner stated the problem than Gauss turned over his slate and exclaimed in his peasant dialect, "Ligget se'." (Here it lies.) For nearly an hour Büttner glared at Gauss, who sat with folded hands while his classmates toiled away. When Büttner examined the slates at the end of the period, Gauss's slate contained a single number, 5050—the only correct solution in the class. To his credit, Büttner recognized the genius of Gauss and with the help of his assistant, John Bartels, had him brought to the attention of Karl Wilhelm Ferdinand, Duke of Brunswick. The shy and awkward boy, who was then fourteen, so captivated the Duke that he subsidized him through preparatory school, college, and the early part of his career.

From 1795 to 1798 Gauss studied mathematics at the University of Göttingen, receiving his degree in absentia from the University of Helmstadt. For his dissertation, he gave the first complete proof of the fundamental theorem of algebra, which states that every polynomial equation has as many solutions as its degree. At age 19 he solved a problem that baffled Euclid, inscribing a regular polygon of 17 sides in a circle using straightedge and compass; and in 1801, at age 24, he published his first masterpiece, *Disquisitiones Arithmeticae*, considered by many to be one of the most brilliant achievements in mathematics. In that book Gauss systematized the study of number theory (properties of the integers) and formulated the basic concepts that form the foundation of that subject.

In the same year that the *Disquisitiones* was published, Gauss again applied his phenomenal computational skills in a dramatic way. The astronomer Giuseppe Piazzi had observed the asteroid Ceres for $\frac{1}{40}$ of its orbit, but lost it in the Sun. Using only three observations and the "method of least squares" that he had developed in 1795, Gauss computed the orbit with such accuracy that astronomers had no trouble relocating it the following year. This achievement brought him instant recognition as the premier mathematician in Europe, and in 1807 he was made Professor of Astronomy and head of the astronomical observatory at Göttingen.

In the years that followed, Gauss revolutionized mathematics by bringing to it standards of precision and rigor undreamed of by his predecessors. He had a passion for perfection that drove him to polish and rework his papers rather than publish less finished work in greater numbers—his favorite saying was "Pauca, sed matura" (Few, but ripe). As a result, many of his important discoveries were squirreled away in diaries that remained unpublished until years after his death.

Among his myriad achievements, Gauss discovered the Gaussian or "bell-shaped" error curve fundamental in probability, gave the first geometric interpretation of complex numbers and established their fundamental role in mathematics, developed methods of characterizing surfaces intrinsically by means of the curves that they contain, developed the theory of conformal (angle-preserving) maps, and discovered non-Euclidean geometry 30 years before the ideas were published by others. In physics he made major contributions to the theory of lenses and capillary action, and with Wilhelm Weber he did fundamental work in electromagnetism. Gauss invented the heliotrope, bifilar magnetometer, and an electrotelegraph.

Gauss was deeply religious and aristocratic in demeanor. He mastered foreign languages with ease, read extensively, and enjoyed mineralogy and botany as hobbies. He disliked teaching and was usually cool and discouraging to other mathematicians, possibly because he had already anticipated their work. It has been said that if Gauss had published all of his discoveries, the current state of mathematics would be advanced by 50 years. He was without a doubt the greatest mathematician of the modern era.

[*Image: ©SSPL/The Image Works*]

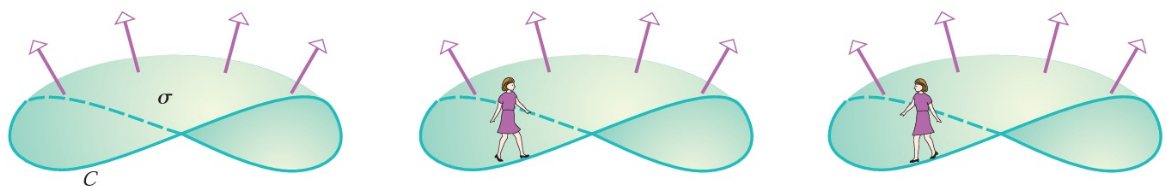
ตัวอย่าง 4.6.5 กำหนดให้ σ เป็นทรงสี่เหลี่ยมที่ล้อมรอบด้วยระนาบ $x+y+z=1$ ที่อยู่ในอัฐภาคที่หนึ่ง กำหนดทิศทางโดยเวกเตอร์แนวฉากหนึ่งหน่วยด้านนอก และให้ $\vec{F}(x,y,z)=x^2\vec{i}+xy\vec{j}+x^3y^3\vec{k}$ จงใช้ทฤษฎีบทไดเวอร์เจนซ์หาค่า

$$\iint_{\sigma} \vec{F}\cdot\vec{n}dS$$

4.7 ทฤษฎีบทของสโตกส์ (Stoke's Theorem)

ในหัวข้อนี้เราจะให้ความสนใจกับพื้นผิวมีทิศทาง (oriented surface) ที่ถูกปิดล้อมด้วยเส้นโค้งปิดอย่างง่ายดังนี้

ให้ σ เป็นพื้นผิวมีทิศทาง (oriented surface) ที่ปิดล้อมด้วยเส้นโค้ง C ซึ่งเป็นเส้นโค้งปิดอย่างง่าย (ดังรูป a) จะพบว่าความสัมพันธ์ของ σ และ C มีได้ 2 ลักษณะ กล่าวคือ



The oriented surface σ is bounded by the simple closed curve C.

(a)

The positive direction of C relative to the orientation of σ.

(b)

The negative direction of C relative to the orientation of σ.

(c)

สมมติให้ น.ส. A เดินไปตามเส้นโค้ง C โดยที่ศีรษะของเธออยู่ในทิศทางเดียวกับทิศทางของ σ เราจะกล่าวว่า

(1) น.ส. A เดินไปในทิศทางบวก (positive direction) ของ C ถ้า พื้นผิว σ อยู่ทางด้านซ้ายมือของ น.ส. A (ดังรูป b)

และ

(2) น.ส. A เดินไปในทิศทางลบ (negative direction) ของ C ถ้า พื้นผิว σ อยู่ทางด้านขวามือของ น.ส. A (ดังรูป c)

ทฤษฎีบทของสโตกส์ : General Form

ให้ σ เป็นพื้นผิวมีทิศทาง (oriented) และปรับเรียบเป็นส่วน ๆ (piecewise smooth) และมีขอบ C เป็นเส้นโค้งอย่างง่าย (simple) ปิด (closed) และปรับเรียบเป็นส่วน ๆ (piecewise smooth) ซึ่งมีทิศทางบวก (positive orientation)

ถ้าฟังก์ชันส่วนประกอบของ

$$\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$$

เป็นฟังก์ชันต่อเนื่อง และอนุพันธ์ย่อยอันดับหนึ่งของฟังก์ชันส่วนประกอบดังกล่าวเป็นฟังก์ชันต่อเนื่องบนบางเซตเปิดที่บรรจุ σ

ถ้า $\mathbf{T}$ เป็นเวกเตอร์สัมผัสหนึ่งหน่วย (unit tangent vector) ของ C แล้ว

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_\sigma (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$

ทั้งนี้ถ้า $\mathbf{F}$ เป็นแรง (force field) จะพบว่า เราสามารถหางาน (work) ของการเคลื่อนที่อนุภาค ด้วยแรง $\mathbf{F}$ ในทิศทางของเส้นโค้ง C ได้ดังนี้

ทฤษฎีบทของสโตกส์: งาน!

ให้ σ เป็นพื้นผิวมีทิศทางปรับเรียบเป็นช่วง ๆ ซึ่งมีขอบ C เป็นเส้นโค้งอย่างง่าย ปิด และปรับเรียบที่มีทิศทางบวก
ถ้าฟังก์ชันส่วนประกอบของ

$$\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$$

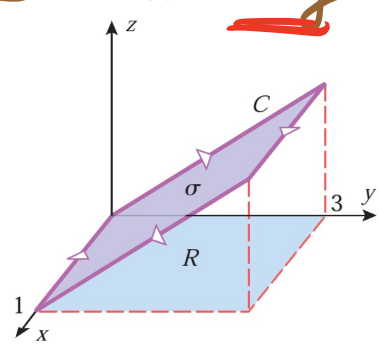
และอนุพันธ์ของฟังก์ชันส่วนประกอบมีเป็นฟังก์ชันต่อเนื่องบนเซตเปิดที่บรรจุ σ แล้ว งาน (work) ของการเคลื่อนที่อนุภาค ด้วยแรง F ไปตาม C หาได้จาก

$$W = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_\sigma (\text{curl}\,\mathbf{F})\cdot \mathbf{n}\, dS$$

ตัวอย่าง: จงหางานที่เกิดจากแรง

$$\checkmark\ \mathbf{F}(x,y,z) = x^2\mathbf{i} + 4xy^3\mathbf{j} + y^2x\mathbf{k}\ \checkmark$$

เพื่อเคลื่อนที่อนุภาคไปตามรูปสี่เหลี่ยม C บนระนาบ $z=y$ ดังภาพ



วิธีทำ STOKE's THM?

σ : simple + piecewise smooth surface

และ C : simple + closed + piecewise smooth curve

และ $\mathbf{F}$: continuous

$$\Rightarrow W = \iint_\sigma (\text{curl}\,\mathbf{F})\cdot \mathbf{n}\, dS = \iint_R (\text{curl}\,\mathbf{F})\cdot \nabla G\, dA$$

① คำนวณ curl $\mathbf{F}$;

$$\operatorname{curl} \mathbb{F} = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & 4xy^3 & y^2x & x^2 & 4xy^3 \end{array} \right.$$

$$\begin{aligned} &= 2xyi + 0\cdot j + 4y^3k \\ &\quad - 0\cdot k - 0\cdot i - y^2 j \\ &= 2xyi - y^2j + 4y^3k \end{aligned}$$

② หา ∇G; จากสมการพื้นผิว $z = y$

นั่นคือ $G(x,y,z) = z - y$

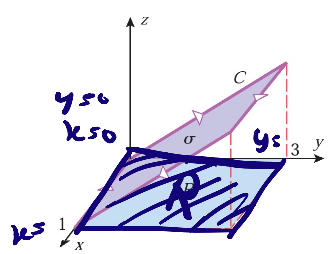
$$\Rightarrow \quad \nabla G = \frac{\partial G}{\partial x} i + \frac{\partial G}{\partial y} j + \frac{\partial G}{\partial z} k$$

$$= 0\cdot i - j + k = -j + k$$

เนื่องจาก C มีทิศทางลบ จะได้ว่า $-\nabla G = -(-j+k) = j - k$

③ หาขอบเขต R ;

$0 \le x \le 1$ และ $0 \le y \le 3$

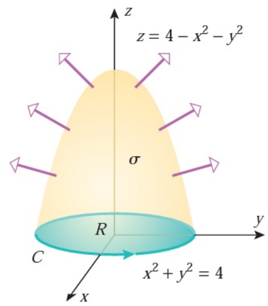


④ หางาน (W);

$$W = \iint_{\sigma} (\operatorname{curl} \mathbb{F}) \cdot n \, dS = \iint_{R} (\operatorname{curl} \mathbb{F}) \cdot (-\nabla G) \, dA$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=1} (2xyi - y^2j + 4y^3k) \cdot (j - k) \, dx\,dy$$

$$\boxed{\int_{y=0}^{y=3} \int_{x=0}^{x=1} (-y^2 - 4y^3) \, dx\,dy}$$



▲ **Figure 15.8.3**

▶ Example 2 Verify Stokes' Theorem for the vector field $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and C to be the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane (Figure 15.8.3).

Solution. We will verify Formula (3). Since σ is oriented up, the positive orientation of C is counterclockwise looking down the positive z-axis. Thus, C can be represented parametrically (with positive orientation) by

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 0 \qquad (0 \leq t \leq 2\pi) \tag{4}$$

Therefore,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C 2z\,dx + 3x\,dy + 5y\,dz\\ &= \int_0^{2\pi} [0 + (6\cos t)(2\cos t) + 0]\,dt\\ &= \int_0^{2\pi} 12\cos^2 t\,dt = 12\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 12\pi\end{aligned}$$

To evaluate the right side of (3), we start by finding curl $\mathbf{F}$. We obtain

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

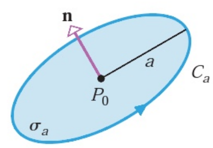
Since σ is oriented up and is expressed in the form $z = g(x, y) = 4 - x^2 - y^2$, it follows from Formula (12) of Section 15.6 with curl $\mathbf{F}$ replacing $\mathbf{F}$ that

$$\begin{aligned}\iint_\sigma (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS &= \iint_R (\operatorname{curl}\mathbf{F})\cdot\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)dA\\ &= \iint_R (5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA\\ &= \iint_R (10x + 4y + 3)\,dA\\ &= \int_0^{2\pi}\int_0^2 (10r\cos\theta + 4r\sin\theta + 3)r\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{10r^3}{3}\cos\theta + \frac{4r^3}{3}\sin\theta + \frac{3r^2}{2}\right]_{r=0}^2 d\theta\\ &= \int_0^{2\pi}\left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right)d\theta\\ &= \left[\frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta\right]_0^{2\pi} = 12\pi\end{aligned}$$

As guaranteed by Stokes' Theorem, the value of this surface integral is the same as the value obtained for the line integral. Note, however, that the line integral was simpler to evaluate and hence would be the method of choice in this case. ◀

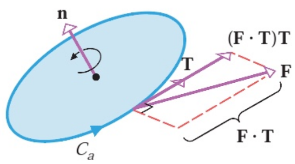
CURL VIEWED AS CIRCULATION

Stokes' Theorem provides a way of interpreting the curl of a vector field $\mathbf{F}$ in the context of fluid flow. For this purpose let σ_a be a small oriented disk of radius a centered at a point P_0 in a steady-state fluid flow, and let $\mathbf{n}$ be a unit normal vector at the center of the disk that points in the direction of orientation. Let us assume that the flow of liquid past the disk causes it to spin around the axis through $\mathbf{n}$, and let us try to find the direction of $\mathbf{n}$ that will produce the maximum rotation rate in the positive direction of the boundary curve C_a (Figure 15.8.4). For convenience, we will denote the area of the disk σ_a by $A(\sigma_a)$; that is, $A(\sigma_a) = \pi a^2$.



▲ **Figure 15.8.4**

If the direction of $\mathbf{n}$ is fixed, then at each point of C_a the only component of $\mathbf{F}$ that contributes to the rotation of the disk about $\mathbf{n}$ is the component $\mathbf{F}\cdot\mathbf{T}$ tangent to C_a (Figure 15.8.5). Thus, for a fixed $\mathbf{n}$ the integral



▲ **Figure 15.8.5**

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{7}$$

can be viewed as a measure of the tendency for the fluid to flow in the positive direction around C_a. Accordingly, (7) is called the ***circulation of*** $\mathbf{F}$ ***around*** $\boldsymbol{C_a}$. For example, in the extreme case where the flow is normal to the circle at each point, the circulation around C_a is zero, since $\mathbf{F}\cdot\mathbf{T} = 0$ at each point. The more closely that $\mathbf{F}$ aligns with $\mathbf{T}$ along the circle, the larger the value of $\mathbf{F}\cdot\mathbf{T}$ and the larger the value of the circulation.

To see the relationship between circulation and curl, suppose that curl $\mathbf{F}$ is continuous on σ_a, so that when σ_a is small the value of curl $\mathbf{F}$ at any point of σ_a will not vary much from the value of curl $\mathbf{F}(P_0)$ at the center. Thus, for a small disk σ_a we can reasonably approximate curl $\mathbf{F}$ on σ_a by the constant value curl $\mathbf{F}(P_0)$. Moreover, because the surface σ_a is flat, the unit normal vectors that orient σ_a are all equal. Thus, the vector quantity $\mathbf{n}$ in Formula (3) can be treated as a constant, and we can write

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\sigma_a} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS \approx \text{curl }\mathbf{F}(P_0)\cdot\mathbf{n}\iint_{\sigma_a} dS$$

where the line integral is taken in the positive direction of C_a. But the last double integral in this equation represents the surface area of σ_a, so

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \approx [\text{curl }\mathbf{F}(P_0)\cdot\mathbf{n}]A(\sigma_a)$$

from which we obtain

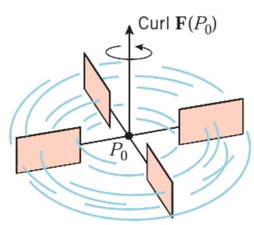
$$\text{curl }\mathbf{F}(P_0)\cdot\mathbf{n} \approx \frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{8}$$

The quantity on the right side of (8) is called the ***circulation density of*** $\mathbf{F}$ ***around*** $\boldsymbol{C_a}$. If we now let the radius a of the disk approach zero (with $\mathbf{n}$ fixed), then it is plausible that the error in this approximation will approach zero and the exact value of curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ will be given by

Formula (9) is sometimes taken as a definition of curl. This is a useful alternative to Definition 15.1.5 because it does not require a coordinate system.

$$\text{curl }\mathbf{F}(P_0)\cdot\mathbf{n} = \lim_{a\to 0}\frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{9}$$

We call curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ the ***circulation density of*** $\mathbf{F}$ ***at*** $\boldsymbol{P_0}$ ***in the direction of*** $\mathbf{n}$. This quantity has its maximum value when $\mathbf{n}$ is in the same direction as curl $\mathbf{F}(P_0)$; this tells us that *at each point in a steady-state fluid flow the maximum circulation density occurs in the direction of the curl*. Physically, this means that if a small paddle wheel is immersed in the fluid so that the pivot point is at P_0, then the paddles will turn most rapidly when the spindle is aligned with curl $\mathbf{F}(P_0)$ (Figure 15.8.6). If curl $\mathbf{F} = \mathbf{0}$ at each point of a region, then $\mathbf{F}$ is said to be ***irrotational*** in that region, since no circulation occurs about any point of the region.



▲ **Figure 15.8.6**

George Gabriel Stokes (1819–1903) Irish mathematician and physicist. Born in Skreen, Ireland, Stokes came from a family deeply rooted in the Church of Ireland. His father was a rector, his mother the daughter of a rector, and three of his brothers took holy orders. He received his early education from his father and a local parish clerk. In 1837, he entered Pembroke College and after graduating with top honors accepted a fellowship at the college. In 1847 he was appointed Lucasian professor of mathematics at Cambridge, a position once held by Isaac Newton (and now held by the British physicist, Stephen Hawking), but one that had lost its esteem through the years. By virtue of his accomplishments, Stokes ultimately restored the position to the eminence it once held. Unfortunately, the position paid very little and Stokes was forced to teach at the Government School of Mines during the 1850s to supplement his income.

Stokes was one of several outstanding nineteenth century scientists who helped turn the physical sciences in a more empirical direction. He systematically studied hydrodynamics, elasticity of solids, behavior of waves in elastic solids, and diffraction of light. For Stokes, mathematics was a tool for his physical studies. He wrote classic papers on the motion of viscous fluids that laid the foundation for modern hydrodynamics; he elaborated on the wave theory of light; and he wrote papers on gravitational variation that established him as a founder of the modern science of geodesy.

Stokes was honored in his later years with degrees, medals, and memberships in foreign societies. He was knighted in 1889. Throughout his life, Stokes gave generously of his time to learned societies and readily assisted those who sought his help in solving problems. He was deeply religious and vitally concerned with the relationship between science and religion.

[Image: http://commons.wikimedia.org/wiki/File:Stokes_George_G.jpg]

✔ QUICK CHECK EXERCISES 15.8 *(See page 1166 for answers.)*

1. Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the component functions of the vector field $\mathbf{F}(x, y, z)$ have continuous first partial derivatives on some open set containing σ, and if $\mathbf{T}$ is the unit tangent vector to C, then Stokes' Theorem states that the line integral ________ and the surface integral ________ are equal.

2. We showed in Example 2 that the vector field

$$\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$$

satisfies the equation curl $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. It follows from Stokes' Theorem that if C is any circle of radius a in the xy-plane that is oriented counterclockwise when viewed from the positive z-axis, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \underline{\hspace{2cm}}$$

where $\mathbf{T}$ denotes the unit tangent vector to C.

3. (a) If σ_1 and σ_2 are two oriented surfaces that have the same positively oriented boundary curve C, and if the vector field $\mathbf{F}(x, y, z)$ has continuous first partial derivatives on some open set containing σ_1 and σ_2, then it follows from Stokes' Theorem that the surface integrals ________ and ________ are equal.

(b) Let $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, let a be a positive number, and let σ be the portion of the paraboloid $z = a^2 - x^2 - y^2$ for which $z \geq 0$ with upward orientation. Using part (a) and Quick Check Exercise 2, it follows that

$$\iint_\sigma (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dS = \underline{\hspace{2cm}}$$

4. For steady-state flow, the maximum circulation density occurs in the direction of the ________ of the velocity vector field for the flow.

EXERCISE SET 15.8 [C] CAS

1–4 Verify Formula (2) in Stokes' Theorem by evaluating the line integral and the surface integral. Assume that the surface has an upward orientation. ■

1. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$; σ is the portion of the plane $x + y + z = 1$ in the first octant.

2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$.

3. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

4. $\mathbf{F}(x, y, z) = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$; σ is the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.

5–12 Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. ■

5. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2x\mathbf{j} - y^3\mathbf{k}$; C is the circle $x^2 + y^2 = 1$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.

6. $\mathbf{F}(x, y, z) = xz\mathbf{i} + 3x^2y^2\mathbf{j} + yx\mathbf{k}$; C is the rectangle in the plane $z = y$ shown in Figure 15.8.2.

7. $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$; C is the boundary of the paraboloid shown in Figure 15.8.3.

8. $\mathbf{F}(x, y, z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$; C is the triangle in the plane $z = \frac{1}{2}y$ with vertices $(2, 0, 0)$, $(0, 2, 1)$, and $(0, 0, 0)$ with a counterclockwise orientation looking down the positive z-axis.

9. $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$; C is the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$ with a counterclockwise orientation looking down the positive z-axis.

10. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$; C is the triangle in the plane $x + y + z = 1$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with a counterclockwise orientation looking from the first octant toward the origin.

11. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$; C is the circle $x^2 + y^2 = a^2$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.

12. $\mathbf{F}(x, y, z) = (z + \sin x)\mathbf{i} + (x + y^2)\mathbf{j} + (y + e^z)\mathbf{k}$; C is the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$ with counterclockwise orientation looking down the positive z-axis.

13–16 True–False Determine whether the statement is true or false. Explain your answer.

13. Stokes' Theorem equates a line integral and a surface integral.

14. Stokes' Theorem is a special case of Green's Theorem.

15. The circulation of a vector field $\mathbf{F}$ around a closed curve C is defined to be
$$\int_C (\text{curl } \mathbf{F}) \cdot \mathbf{T}\, ds$$

16. If $\mathbf{F}(x, y, z)$ is defined everywhere in 3-space, and if curl $\mathbf{F}$ has no $\mathbf{k}$-component at any point in the xy-plane, then
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$$
for every smooth, simple, closed curve in the xy-plane.

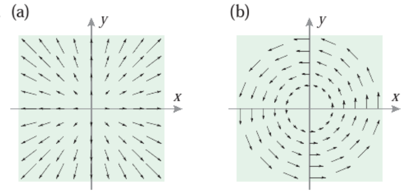
17. Consider the vector field given by the formula
$$\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - xy)\mathbf{k}$$
(a) Use Stokes' Theorem to find the circulation around the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, and $C(0, 0, 1)$ oriented counterclockwise looking from the origin toward the first octant.
(b) Find the circulation density of $\mathbf{F}$ at the origin in the direction of $\mathbf{k}$.
(c) Find the unit vector $\mathbf{n}$ such that the circulation density of $\mathbf{F}$ at the origin is maximum in the direction of $\mathbf{n}$.

FOCUS ON CONCEPTS

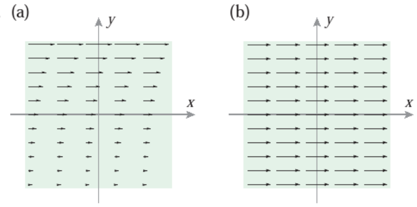
18. (a) Let σ denote the surface of a solid G with $\mathbf{n}$ the outward unit normal vector field to σ. Assume that $\mathbf{F}$ is a vector field with continuous first-order partial derivatives on σ. Prove that
$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$
[*Hint:* Let C denote a simple closed curve on σ that separates the surface into two subsurfaces σ_1 and σ_2 that share C as their common boundary. Apply Stokes' Theorem to σ_1 and to σ_2 and add the results.]
(b) The vector field curl($\mathbf{F}$) is called the ***curl field*** of $\mathbf{F}$. In words, interpret the formula in part (a) as a statement about the flux of the curl field.

19–20 The figures in these exercises show a horizontal layer of the vector field of a fluid flow in which the flow is parallel to the xy-plane at every point and is identical in each layer (i.e., is independent of z). For each flow, state whether you believe that the curl is nonzero at the origin, and explain your reasoning. If you believe that it is nonzero, then state whether it points in the positive or negative z-direction.

19. (a) (b)



20. (a) (b)



21. Let $\mathbf{F}(x, y, z)$ be a conservative vector field in 3-space whose component functions have continuous first partial derivatives. Explain how to use Formula (9) to prove that curl $\mathbf{F} = \mathbf{0}$.

22. In 1831 the physicist Michael Faraday discovered that an electric current can be produced by varying the magnetic flux through a conducting loop. His experiments showed that the electromotive force $\mathbf{E}$ is related to the magnetic induction $\mathbf{B}$ by the equation
$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\iint_\sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS$$
Use this result to make a conjecture about the relationship between curl $\mathbf{E}$ and $\mathbf{B}$, and explain your reasoning.

C 23. Let σ be the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$, and let C be the circle $x^2 + y^2 = 1$ that forms the boundary of σ in the xy-plane. Assuming that σ is oriented up, use a CAS to verify Formula (2) in Stokes' Theorem for the vector field
$$\mathbf{F} = (x^2y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$
by evaluating the line integral and the surface integral.

24. **Writing** Discuss what it means to say that the curl of a vector field is independent of a coordinate system. Explain how we know this to be true.

25. **Writing** Compare and contrast the Fundamental Theorem of Line Integrals, the Divergence Theorem, and Stokes' Theorem.

✔ QUICK CHECK ANSWERS 15.8

1. $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; $\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ **2.** $3\pi a^2$ **3.** (a) $\iint_{\sigma_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$; $\iint_{\sigma_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ (b) $3\pi a^2$ **4.** curl