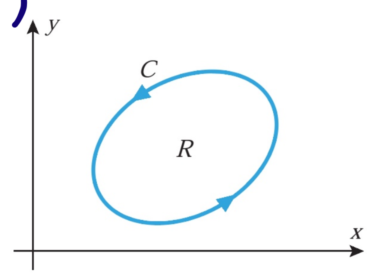
4.3 ทฤษฎีบทของกรีน (Green's Theorem)

ทฤษฎีบทของกรีน :

ให้ R เป็นบริเวณเชื่อมโยงอย่างง่ายที่ขอบเป็นเส้นโค้ง C ซึ่งเป็นเส้นโค้งอย่างง่าย (simple) ปิด (closed) ปรับเรียบเป็นช่วงๆ (piecewise smooth) และมีทิศทางทวนเข็มนาฬิกา (counter clockwise)

ถ้า $f(x,y)$ และ $g(x,y)$ เป็นฟังก์ชันต่อเนื่อง และมีอนุพันธ์ย่อยอันดับหนึ่งที่ต่อเนื่องบนบางบริเวณปิดที่บรรจุ R แล้ว

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$



George Green **(1793–1841)** English mathematician and physicist. Green left school at an early age to work in his father's bakery and consequently had little early formal education. When his father opened a mill, the boy used the top room as a study in which he taught himself physics and mathematics from library books. In 1828 Green published his most important work, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Although Green's Theorem appeared in that paper, the result went virtually unnoticed because of the small pressrun and local distribution. Following the death of his father in 1829, Green was urged by friends to seek a college education. In 1833, after four years of self-study to close the gaps in his elementary education, Green was admitted to Caius College, Cambridge. He graduated four years later, but with a disappointing performance on his final examinations—possibly because he was more interested in his own research. After a succession of works on light and sound, he was named to be Perse Fellow at Caius College. Two years later he died. In 1845, four years after his death, his paper of 1828 was published and the theories developed therein by this obscure, self-taught baker's son helped pave the way to the modern theories of electricity and magnetism.

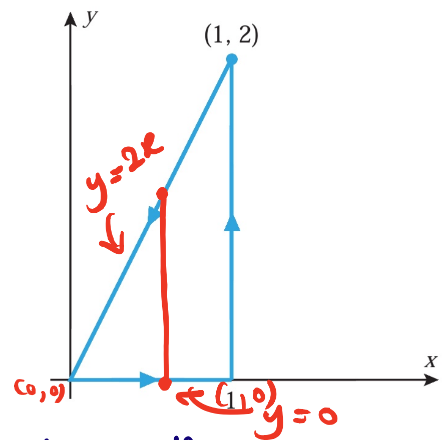
หมายเหตุ : เราอาจใช้สัญลักษณ์แทนการหาปริพันธ์ตามเส้นรอบเส้นโค้งอย่างง่ายปิดดังนี้

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

ตัวอย่าง : จงใช้ทฤษฎีบทของกรีนหาค่าของ

$$\oint_C \underbrace{x^2y}_{f(x,y)}\,dx + \underbrace{x}_{g(x,y)}\,dy$$

โดยที่ C เป็นวิถีสามเหลี่ยมดังรูป



วิธีทำ ① Check! C : simple, closed, piecewise smooth curve

Check! $f(x,y)$: cont. และ $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$: cont. ✓

และ $g(x,y)$: cont. และ $\frac{\partial g}{\partial x} = 1$, $\frac{\partial g}{\partial y} = 0$: cont. ✓

ดังนั้น เราประยุกต์ใช้ทฤษฎีบทของกรีน จะได้ว่า

หาสมการ (line) :

$m = \frac{2-0}{1-0} = 2$

$\Rightarrow y - 0 = 2(x-0)$

$\Rightarrow y = 2x$

$$\oint_C x^2y\,dx + x\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=2x} (1 - x^2)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left[y - yx^2\right]_{y=0}^{y=2x} dx$$

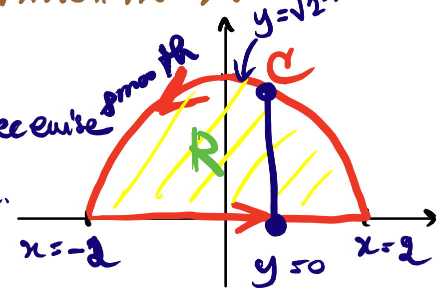
$$= \int_{x=0}^{x=1} \left[2x - 2x^3\right] dx$$

$$= \left[x^2 - \frac{1}{2}x^4\right]_{x=0}^{x=1} = 1^2 - \frac{1}{2}(1)^4 = \frac{1}{2} \quad \square$$

ตัวอย่าง: จงใช้ทฤษฎีบทของกรีนหาค่า

$$\oint_C (e^{x^2}-y)\,dx + (x+\sin\sqrt{y})\,dy$$

เมื่อ C เป็นวิถีครึ่งวงกลมรัศมี 2 หน่วยเหนือแกน X



วิธีทำ (ผ่าน!) Check! C: simple, closed, piecewise smooth

f, g : cont. และ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$: cont.

จาก $f(x,y) = e^{x^2}-y \Rightarrow \frac{\partial f}{\partial y} = 0-1 = -1$

และ $g(x,y) = x+\sin\sqrt{y} \Rightarrow \frac{\partial g}{\partial x} = 1-0 = 1$

โดยใช้การประยุกต์ใช้ทฤษฎีบทของกรีนจะได้ว่า

$$\oint_C (e^{x^2}-y)\,dx + (x+\sin\sqrt{y})\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \int_{x=-2}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} \left(1-(-1)\right) dy\,dx$$

$$= \int_{x=-2}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} 2\,dy\,dx$$

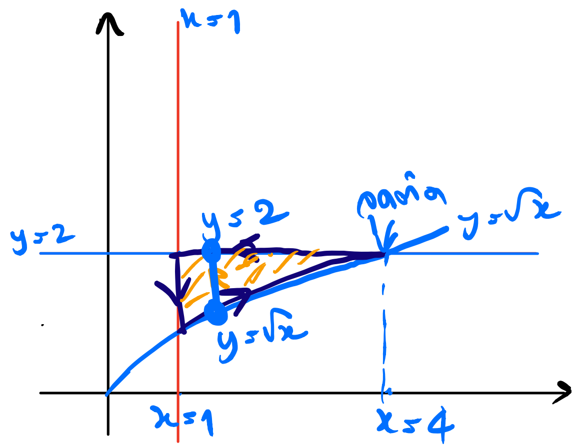
$$= \cdots$$

□

ตัวอย่าง : จงใช้ทฤษฎีบทของกรีนหาค่า

$$\oint_C (y^2+\sin x)\,dx + (xy+\ln x)\,dy$$

เมื่อ C เป็นวิถีล้อมรอบเส้นตรง $x=1$, $y=2$ และเส้นโค้ง $y=\sqrt{x}$



$f(x,y) = y^2+\sin x$

$\Rightarrow \dfrac{\partial f}{\partial y} = 2y$

และ $g(x,y) = xy+\ln x$

$\Rightarrow \dfrac{\partial g}{\partial x} = y + \dfrac{1}{x}$

ดังนั้น การประยุกต์ใช้ทฤษฎีบทของกรีน จะได้ว่า

$$\oint_C (y^2+\sin x)\,dx + (xy+\ln x)\,dy$$

$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int_{x=1}^{x=4}\int_{y=\sqrt{x}}^{y=2} \left(y+\frac{1}{x}-2y\right) dy\,dx$$

$$= \cdots$$

ทั้งนี้เราสามารถประยุกต์ใช้กฎของกรีนในการหาพื้นที่ A ของบริเวณ R ซึ่งมีสมบัติสอดคล้องกับเงื่อนไขในกฎของกรีน โดยสูตร

$$A = \iint_R dA = \oint_C x\,dy \qquad \text{—— ①}$$

หรือ

$$A = \iint_R dA = \oint_C (-y)\,dx \qquad \text{—— ②}$$

และ: ถ้านำ ① และ ② มารวมกันจะได้

$$\boxed{A = \frac{1}{2}\oint_C (-y)\,dx + x\,dy}$$

ใช้สูตรนี้!

ตัวอย่าง: จงใช้ปริพันธ์ตามเส้นหาพื้นที่รูปสามเหลี่ยมที่มีจุดยอดเป็น (0,0), (a,0) และ (0,b) โดยที่ a,b > 0

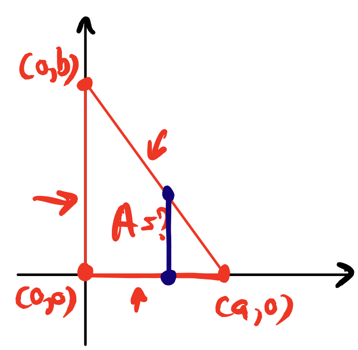
$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$

g(x,y), f(x,y) = 0

$$A = \oint_C x\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=-\frac{bx}{a}+b} (1-0)\,dy\,dx$$

$$= \dots = ab$$



$$m = \frac{b-0}{0-a} = -\frac{b}{a}$$

$$\Rightarrow y - 0 = -\frac{b}{a}(x-a)$$

$$\Rightarrow y = -\frac{bx}{a} + b$$

✔ QUICK CHECK EXERCISES 15.4

(See page 1129 for answers.)

1. If C is the square with vertices $(\pm 1, \pm 1)$ oriented counterclockwise, then

$$\int_C -y\,dx + x\,dy = \_\_\_\_\_\_$$

2. If C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$ oriented counterclockwise, then

$$\int_C 2xy\,dx + (x^2 + x)\,dy = \_\_\_\_\_\_$$

3. If C is the unit circle centered at the origin and oriented counterclockwise, then

$$\int_C (y^3 - y - x)\,dx + (x^3 + x + y)\,dy = \_\_\_\_\_\_$$

4. What region R and choice of functions $f(x, y)$ and $g(x, y)$ allow us to use Formula (1) of Theorem 15.4.1 to claim that

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (2x + 2y)\,dy\,dx = \int_0^{\pi/2} (\sin^3 t + \cos^3 t)\,dt?$$

EXERCISE SET 15.4 C CAS

1–2 Evaluate the line integral using Green's Theorem and check the answer by evaluating it directly. ■

1. $\oint_C y^2\,dx + x^2\,dy$, where C is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ oriented counterclockwise.

2. $\oint_C y\,dx + x\,dy$, where C is the unit circle oriented counterclockwise.

3–13 Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise. ■

3. $\oint_C 3xy\,dx + 2xy\,dy$, where C is the rectangle bounded by $x = -2$, $x = 4$, $y = 1$, and $y = 2$.

4. $\oint_C (x^2 - y^2)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.

5. $\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, and $(0, \pi/2)$.

6. $\oint_C y\tan^2 x\,dx + \tan x\,dy$, where C is the circle $x^2 + (y+1)^2 = 1$.

7. $\oint_C (x^2 - y)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 4$.

8. $\oint_C (e^x + y^2)\,dx + (e^y + x^2)\,dy$, where C is the boundary of the region between $y = x^2$ and $y = x$.

9. $\oint_C \ln(1+y)\,dx - \dfrac{xy}{1+y}\,dy$, where C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

10. $\oint_C x^2 y\,dx - y^2 x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle $x^2 + y^2 = 16$.

11. $\oint_C \tan^{-1} y\,dx - \dfrac{y^2 x}{1+y^2}\,dy$, where C is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

12. $\oint_C \cos x \sin y\, dx + \sin x \cos y\, dy$, where C is the triangle with vertices $(0, 0)$, $(3, 3)$, and $(0, 3)$.

13. $\oint_C x^2 y\, dx + (y + xy^2)\, dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

14. Let C be the boundary of the region enclosed between $y = x^2$ and $y = 2x$. Assuming that C is oriented counterclockwise, evaluate the following integrals by Green's Theorem:

(a) $\oint_C (6xy - y^2)\, dx$ (b) $\oint_C (6xy - y^2)\, dy$.

15–18 True–False Determine whether the statement is true or false. Explain your answer. (In Exercises 16–18, assume that C is a simple, smooth, closed curve, oriented counterclockwise.)

15. Green's Theorem allows us to replace any line integral by a double integral.

16. If

$$\int_C f(x, y)\, dx + g(x, y)\, dy = 0$$

then $\partial g/\partial x = \partial f/\partial y$ at all points in the region bounded by C.

17. It must be the case that

$$\int_C x\, dy > 0$$

18. It must be the case that

$$\int_C e^{x^2} dx + \sin y^3\, dy = 0$$

C **19.** Use a CAS to check Green's Theorem by evaluating both integrals in the equation

$$\oint_C e^y\, dx + ye^x\, dy = \iint_R \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(e^y)\right] dA$$

where

(a) C is the circle $x^2 + y^2 = 1$

(b) C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

20. In Example 3, we used Green's Theorem to obtain the area of an ellipse. Obtain this area using the first and then the second formula in (6).

21. Use a line integral to find the area of the region enclosed by the astroid

$$x = a\cos^3\phi, \quad y = a\sin^3\phi \quad (0 \le \phi \le 2\pi)$$

22. Use a line integral to find the area of the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$.

23. Use the formula

$$A = \frac{1}{2}\oint_C -y\, dx + x\, dy$$

to find the area of the region swept out by the line from the origin to the ellipse $x = a\cos t$, $y = b\sin t$ if t varies from $t = 0$ to $t = t_0$ $(0 \le t_0 \le 2\pi)$.

24. Use the formula

$$A = \frac{1}{2}\oint_C -y\, dx + x\, dy$$

to find the area of the region swept out by the line from the origin to the hyperbola $x = a\cosh t$, $y = b\sinh t$ if t varies from $t = 0$ to $t = t_0$ $(t_0 \ge 0)$.

FOCUS ON CONCEPTS

25. Suppose that $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a vector field whose component functions f and g have continuous first partial derivatives. Let C denote a simple, closed, piecewise smooth curve oriented counterclockwise that bounds a region R contained in the domain of $\mathbf{F}$. We can think of $\mathbf{F}$ as a vector field in 3-space by writing it as

$$\mathbf{F}(x, y, z) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j} + 0\mathbf{k}$$

With this convention, explain why

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\, dA$$

26. Suppose that $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a vector field on the xy-plane and that f and g have continuous first partial derivatives with $f_y = g_x$ everywhere. Use Green's Theorem to explain why

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

where C_1 and C_2 are the oriented curves in the accompanying figure. [*Note:* Compare this result with Theorems 15.3.2 and 15.3.3.]

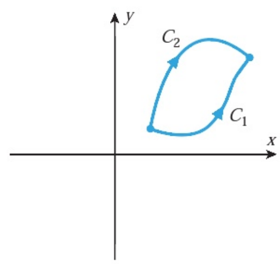


Figure Ex-26

27. Suppose that $f(x)$ and $g(x)$ are continuous functions with $g(x) \le f(x)$. Let R denote the region bounded by the graph of f, the graph of g, and the vertical lines $x = a$ and $x = b$. Let C denote the boundary of R oriented counterclockwise. What familiar formula results from applying Green's Theorem to $\int_C (-y)\, dx$?

28. In the accompanying figure on the next page, C is a smooth oriented curve from $P(x_0, y_0)$ to $Q(x_1, y_1)$ that is contained inside the rectangle with corners at the origin and Q and outside the rectangle with corners at the origin and P.

(a) What region in the figure has area $\int_C x\, dy$?

(b) What region in the figure has area $\int_C y\, dx$?

(c) Express $\int_C x\, dy + \int_C y\, dx$ in terms of the coordinates of P and Q.

(*cont.*)

(d) Interpret the result of part (c) in terms of the Fundamental Theorem of Line Integrals.

(e) Interpret the result in part (c) in terms of integration by parts.

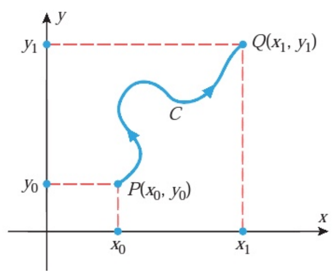


Figure Ex-28

29–30 Use Green's Theorem to find the work done by the force field **F** on a particle that moves along the stated path. ■

29. $\mathbf{F}(x, y) = xy\mathbf{i} + \left(\frac{1}{2}x^2 + xy\right)\mathbf{j}$; the particle starts at $(5, 0)$, traverses the upper semicircle $x^2 + y^2 = 25$, and returns to its starting point along the x-axis.

30. $\mathbf{F}(x, y) = \sqrt{y}\,\mathbf{i} + \sqrt{x}\,\mathbf{j}$; the particle moves counterclockwise one time around the closed curve given by the equations $y = 0$, $x = 2$, and $y = x^3/4$.

31. Evaluate $\oint_C y\,dx - x\,dy$, where C is the cardioid

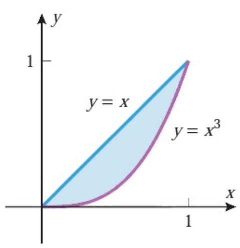
$$r = a(1 + \cos\theta) \quad (0 \le \theta \le 2\pi)$$

32. Let R be a plane region with area A whose boundary is a piecewise smooth, simple, closed curve C. Use Green's Theorem to prove that the centroid $(\bar{x}, \bar{y})$ of R is given by

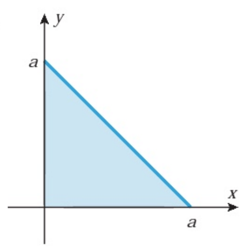
$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy, \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

33–36 Use the result in Exercise 32 to find the centroid of the region. ■

33.



34.



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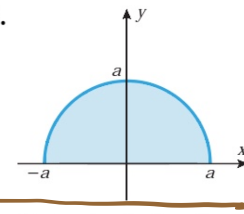

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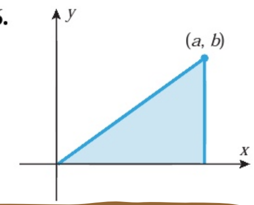

37. Find a simple closed curve C with counterclockwise orientation that maximizes the value of

$$\oint_C \tfrac{1}{3}y^3\,dx + \left(x - \tfrac{1}{3}x^3\right)dy$$

and explain your reasoning.

38. (a) Let C be the line segment from a point (a, b) to a point (c, d). Show that

$$\int_C -y\,dx + x\,dy = ad - bc$$

(b) Use the result in part (a) to show that the area A of a triangle with successive vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) going counterclockwise is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

(c) Find a formula for the area of a polygon with successive vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ going counterclockwise.

(d) Use the result in part (c) to find the area of a quadrilateral with vertices $(0, 0)$, $(3, 4)$, $(-2, 2)$, $(-1, 0)$.

39–40 Evaluate the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the boundary of the region R and C is oriented so that the region is on the left when the boundary is traversed in the direction of its orientation. ■

39. $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (4x - \cos y)\mathbf{j}$; C is the boundary of the region R that is inside the square with vertices $(0, 0)$, $(5, 0)$, $(5, 5)$, $(0, 5)$ but is outside the rectangle with vertices $(1, 1)$, $(3, 1)$, $(3, 2)$, $(1, 2)$.

40. $\mathbf{F}(x, y) = (e^{-x} + 3y)\mathbf{i} + x\mathbf{j}$; C is the boundary of the region R inside the circle $x^2 + y^2 = 16$ and outside the circle $x^2 - 2x + y^2 = 3$.

41. Writing Discuss the role of the Fundamental Theorem of Calculus in the proof of Green's Theorem.

42. Writing Use the Internet or other sources to find information about "planimeters," and then write a paragraph that describes the relationship between these devices and Green's Theorem.

✔ QUICK CHECK ANSWERS 15.4

1. 8 **2.** $\frac{1}{2}$ **3.** 2π **4.** R is the region $x^2 + y^2 \le 1$ $(0 \le x, 0 \le y)$ and $f(x, y) = -y^2$, $g(x, y) = x^2$.